NOTE ON SIGNATURE CHANGE AND COLOMBEAU THEORY

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Recent work alludes to various 'controversies' associated with signature change in general relativity. As we have argued previously, these are in fact disagreements about the (often unstated) assumptions underlying various possible approaches. The choice between approaches remains open.

I. INTRODUCTION

In a recent paper [1], Mansouri and Nozari (MN) discuss signature change in General Relativity Theory from the point of view of Colombeau's generalised functions. They then claim that this "solves the controversy in the literature" regarding the "vanishing of ... Einstein's equation on a surface of signature change." We argue that there is in fact no mathematically based issue here that can be 'resolved' in this way. Rather, there is no clear choice of what should qualify as 'Einstein's equations' in this generalised context, and several options are open, even when Colombeau theory is used. One can make various mathematical proposals in this regard; which of these, if any, have anything to do with real physics is unclear, and is open to debate. We also point out that a reasonable interpretation of Mansouri and Nozari's equations leads to boundary conditions at a signature change which support the choice made by Dray et al. [2–4] and Ellis and coworkers [5–7] in the original signature change 'controversy' [8–11]. To illustrate the point, we append an example with non-zero extrinsic curvature, contrasting with MN's example that has zero extrinsic curvature at the signature change.

II. EINSTEIN'S EQUATIONS AND BOUNDARY CONDITIONS

It is not possible for a nondegenerate metric to change signature, so signature change inevitably involves degenerate metrics. But Einstein's equations are usually derived under the assumption that the metric is nondegenerate, so attempts to apply the Einstein equations through such a signature change run into problems. Several attempts have been made to address this apparent conflict by making generalisations of the Einstein equations, with varying results. An alternative is simply to apply the standard Einstein equations on each side of the surface where the signature changes, and to impose suitable boundary conditions at that surface.

Thus, in the absence of a clear statement of just what Einstein's equations are in these circumstances, authors have argued in favor of particular boundary conditions, which should hold on a surface of signature change. The two basic choices are for the extrinsic curvature to be continuous but not necessarily zero [2–6], or for it to be continuous and to vanish there [12]. One can also consider discontinuities in the extrinsic curvature [7], but attempts to relate these to a distributional matter source at the boundary require some form of field equations valid on the surface.

Several authors have given derivations of generalised forms of the Einstein's equations in the presence of signature change. Kossowski and Kriele [13] assume that the standard form of Einstein's equations written in a specific way should continue to hold in the presence of signature change, and show that the absence of a surface layer (i.e. a distributional term in the matter at the boundary) then forces the extrinsic curvature to vanish there. Dray [14,15] derives (a different version of) Einstein's equations from a variational principle, and shows that the absence of a surface layer in this case forces the extrinsic curvature to be continuous but not necessarily zero — the Darmois boundary conditions [16]. Thus, each of the basic boundary conditions follows from forms of the Einstein field equations, which in turn can be derived from suitable starting points; different starting points give different results.

Mansouri and Nozari [1] repeat the standard calculation of the Einstein tensor using Colombeau's generalised functions, and claim (equations (33)–(36)) that it is proportional to the discontinuity in the extrinsic curvature.

Thus, also in this approach, the absence of a surface layer at the boundary forces the extrinsic curvature to be continuous, but not necessarily zero. This is made clear by the example in the appendix.

Furthermore, the proportionality factor in the Mansouri and Nozari result for the energy momentum tensor on the change surface (equation (57)) depends on the regularisation used in the Colombeau version of the Dirac delta function at the signature change surface. This is rather awkward, and can only be avoided if the extrinsic curvature is continuous, which implies that this energy-momentum tensor is zero. It thus appears that a reasonable interpretation of the Mansouri and Nozari approach is that it is only possible in a parameterisation invariant way if there is no distributional term in the matter at the boundary, unlike the other derivations just cited, and in apparent contradiction to claims in their paper (page 266).

Finally, although these calculations if anything support our previous position rather than others that have been put forward (indeed the paper appears to be incorrect when it states on page 255 that it supports the result of their reference 18 — our [10]), we record an uneasiness with its methods due to some problems with the 'microstructure' embodied in the Colombeau approach, adopted in order to avoid the usual problems associated with products of distributions. Kamleh [17] has recently given a rigorous treatment of signature change using Colombeau's generalised functions, and concludes that "the Colombeau algebra is ... unable to settle the dispute over the nature of the junction conditions."

III. CONCLUSION

As we have argued elsewhere [18], one must ultimately choose what one means by Einstein's equations in the presence of signature change; this does not follow from the standard equations, which are only valid for a regular metric, and so involves some choice as to how to generalise the usual equations in a way that covers this situation. Different mathematical choices lead to different boundary conditions, and may be relevant to different physical situations. The paper by Mansouri and Nozari [1] gives a derivation of one possible version of Einstein's equations within the framework of Colombeau's generalised functions. Other versions, together with suitable derivations, exist, as shown by Kamleh [17]. Thus, attempts to label any specific choice as 'the' Einstein equations are necessarily based in rhetoric rather than physics or mathematics. Ultimately, one must choose which version one prefers; such a choice is not forced on one by the mathematics of the situation, and what experimental physics may imply is as yet unknown.

The example provided shows that MNs results do not support the view of [8,10,12,13].

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APPENDIX A: CONTRASTING EXAMPLE

We start from the de Sitter metric, eq (59) of MN, and use their methods and notation,

$$ds^2 = -f(t)\,dt^2 + a^2(t)\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)\right) \quad . \tag{A1}$$

In a Euclidean region, where $f < 0$,

$$a_E = \alpha_-\ \cos\left(\frac{t}{\alpha_-}\right) \quad , \qquad \dot a_E = -\sin\left(\frac{t}{\alpha_-}\right) \quad , \tag{A2}$$

and in a Lorentzian region, $f > 0$,

$$a_L = \alpha_+\ \cosh\left(\frac{t}{\alpha_+}\right) \quad , \qquad \dot a_L = \sinh\left(\frac{t}{\alpha_+}\right) \quad . \tag{A3}$$

Now, according to MN's equations (53)-(56), there would be no surface layer at a signature change if $[\dot a] = 0$. As MN say on p268, it remains to be checked whether signature changes at times other than the maximum (minimum) in $a(t)$ on the Euclidean (Lorentzian) side can be free of surface layers. We do that here.

We insert a signature change at some time t_- on the Euclidean side and time t_+ on the Lorentzian side, and then shift the origin of time on both sides to make $t = 0$ there. Specifically,

$$f = -\theta(-t) + \theta(t) \quad , \qquad a^2 = a_E^2(t_- + t)\,\theta(-t) + a_L^2(t_+ + t)\,\theta(t) \quad , \tag{A4}$$

and we have no surface layer at $t = 0$ if

$$a_- = a_E(t_-) = a_- = a_L(t_+) \quad , \qquad \dot a_- = \dot a_E(t_-) = \dot a_+ = \dot a_L(t_+) \quad . \tag{A5}$$

This is achieved by solving

$$\alpha_-\ \cos\left(\frac{t_-}{\alpha_-}\right) = \alpha_+\ \cosh\left(\frac{t_+}{\alpha_+}\right) \qquad \text{and} \qquad -\sin\left(\frac{t_-}{\alpha_-}\right) = \sinh\left(\frac{t_+}{\alpha_+}\right) \quad , \tag{A6}$$

for which a couple of specific numerical solutions are:

$$\text{choosing}\quad \alpha_- = 3 \quad , \qquad t_- = -0.5 \qquad \to \qquad \alpha_+ = 2.918540931 \quad , \qquad t_+ = 0.4819808451 \tag{A7}$$

$$\to \qquad a_\pm = 2.958429695 \quad , \qquad \dot a_\pm = 0.1658961327 \quad ; \tag{A8}$$

$$\text{choosing}\quad \alpha_- = 7 \quad , \qquad t_- = -1 \qquad \to \qquad \alpha_+ = 6.859521334 \quad , \qquad t_+ = 0.9733324138 \tag{A9}$$

$$\to \qquad a_\pm = 6.928692823 \quad , \qquad \dot a_\pm = .1423717298 \quad . \tag{A10}$$

MN's example had $t_- = 0$, $t_+ = 0$, $\alpha_+ = \alpha_-$. Clearly, signature changes without surface layers are possible at any time. This applies not only to the de Sitter case, but almost any RW model, and indeed more generally[1].

For comparison, the Darmois-Israel conditions require continuity of the intrinsic metric and extrinsic curvature on the change surface,

$$h_{ij} = a^2\,\mathrm{diag}(1, \sin^2\chi, \sin^2\chi\ \sin^2\theta) \quad , \qquad K_{ij} = \frac{a\dot a}{\sqrt{|f|}}\,\mathrm{diag}(1, \sin^2\chi, \sin^2\chi\ \sin^2\theta) \quad . \tag{A11}$$

i.e. $[a] = 0$ and $[\dot a/\sqrt{|f|}\ ] = 0$. Provided a continuous a is implicit and $|f_+| = |f_-| > 0$, there is no surface layer, in agreement with the above[2].

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[1] We note that the cosmological constant jumps from $3/\alpha_-^2$ to $3/\alpha_+^2$ in the above example, but this is not a problem, as discontinuities in the matter occur with the usual (Lorentzian to Lorentzian) boundary conditions, whereas at a signature change the whole nature of physics changes and causality suddenly appears. Indeed a jump is a much weaker singularity than a surface layer, which MN allow (top half of p266).

[2] It should be emphasised that the surface effects found in [4] are delta functions in the conservation laws, not in the Einstein/matter tensor.